MATH 1A - MIDTERM 3 - SOLUTIONS

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1. *(25 points)* Sketch a graph of the function $f(x) = x\ln(x) - x$. Your work should include:
   - Domain
   - Intercepts
   - Symmetry
   - Asymptotes (no Slant asymptotes, though)
   - Intervals of increase/decrease/local max/min
   - Concavity and inflection points

(1) Domain: $x > 0$

(2) No $y-$intercepts, $x-$intercept $x = e$ ($f(x) = 0 \Leftrightarrow x\ln(x) - x = 0 \Leftrightarrow x\ln(x) = x \Leftrightarrow \ln(x) = 1 \Leftrightarrow x = e$)

(3) No symmetry

(4) **NO** H.A. or V.A., because:

$$\lim_{x\to\infty} x\ln(x) - x = \lim_{x\to\infty} x(\ln(x) - 1) = \infty \times \infty = \infty$$

and:

$$\lim_{x\to 0^+} x\ln(x) - x = \lim_{x\to 0^+} x(\ln(x) - 1) = \lim_{x\to 0^+} \frac{\ln(x) - 1}{\frac{1}{x}} \stackrel{H}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} -x = 0$$

(H means l'Hopital's rule)

(5) $f'(x) = \ln(x) + 1 - 1 = \ln(x)$, f is decreasing on $(0,1)$ and increasing on $(1,\infty)$, $(1,-1)$ is a local minimum by the first derivative test.

(6) $f''(x) = \frac{1}{x} > 0$ (if $x > 0$), f is concave up on $(0,\infty)$, no inflection points

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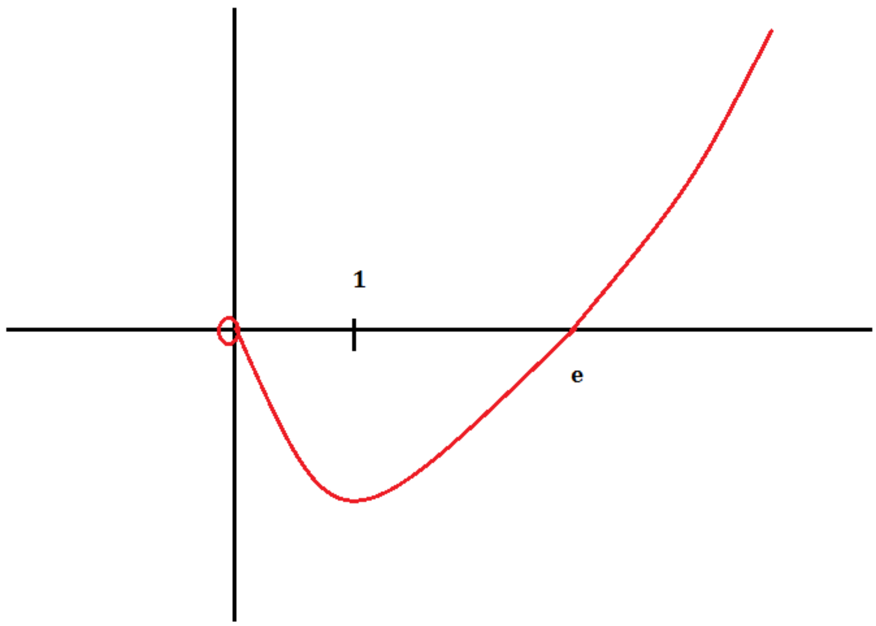
(7) Graph:

1A/Practice Exams/Mockgraph.png



2. *(10 points)* Use a linear approximation (or differentials) to find an approximate value of $\sqrt{99}$

Linear approximation:

Let $f(x) = \sqrt{x}$, $a = 100$. Then $f(100) = 10$ and $f'(100) = \frac{1}{2(10)} = \frac{1}{20}$, so

$$L(x) = f(a) + f'(a)(x - a) = 10 + \frac{1}{20}(x - 100)$$

Now:

$$\sqrt{99} = f(99) \approx L(99) = 10 + \frac{1}{20}(99 - 100) = 10 - 0.05 = 9.95$$

Differentials:

Let $f(x) = \sqrt{x}$, $x = 100$, $dx = 99 - 100 = -1$. Then:

$$dy = f'(x)dx = \frac{1}{20}(-1) = -0.05$$

And

$$\sqrt{99} \approx f(100) + dy = 10 - 0.05 = 9.95$$

3. *(15 points)* Assume the radius of a cone is increasing at a rate of 3 cm/s while its height is decreasing at a rate of 1 cm/s. At what rate is its volume increasing/decreasing when its radius is 2 cm and its volume is $\frac{4\pi}{3}\ cm^3$?

(1) Picture: Just draw a picture of a cone with base radius r and height h.

(2) WTF $\frac{dV}{dt}$ when $r = 2$ and $V = \frac{4\pi}{3}\ cm^3$

(3)
$$V = \frac{\pi}{3} r^2 h$$

(4)
$$\frac{dV}{dt} = \frac{\pi}{3} 2r \frac{dr}{dt} h + \frac{\pi}{3} r^2 \frac{dh}{dt}$$

(5) But $r = 2$, $\frac{dr}{dt} = 3$ and $\frac{dh}{dt} = -1$ (since h is decreasing). Finally, to figure out h, use the fact that $V = \frac{\pi}{3} r^2 h$, so we get:
$$\frac{4\pi}{3} = \frac{\pi}{3} 2^2 h$$
So $h = 1$, and finally, we get:

(6)
$$\frac{dV}{dt} = \frac{\pi}{3} 2(2)(3)(1) + \frac{\pi}{3}(2)^2(-1) = 4\pi - \frac{4}{3}\pi = \frac{8\pi}{3}$$
Whence, $\boxed{\frac{dV}{dt} = \frac{8\pi}{3}\ cm^3/s}$

4. *(10 points)* Find the following limits:

(a)

$$\lim_{x\to 0} x^2 \ln(x) = \lim_{x\to 0} \frac{\ln(x)}{\frac{1}{x^2}} \stackrel{H}{=} \lim_{x\to 0} \frac{\frac{1}{x}}{\frac{-2}{x^3}} = \lim_{x\to 0} -\frac{x^2}{2} = 0$$

(b) $\lim_{x\to\infty} x^{\frac{1}{x}}$

1) Let $y = x^{\frac{1}{x}}$
2) $\ln(y) = \frac{1}{x}\ln(x) = \frac{\ln(x)}{x}$
3)

$$\lim_{x\to\infty} \frac{\ln(x)}{x} \stackrel{H}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$$

4) Hence $\lim_{x\to\infty} x^{\frac{1}{x}} = e^0 = 1$

5. *(10 points)* Find the absolute maximum and minimum of f on $[0, 2]$, where:

$$f(x) = x^4 - 4x + 1$$

1) Endpoints: $f(0) = 1$, $f(2) = 16 - 8 + 1 = 9$

2) Critical numbers:

$$f'(x) = 4x^3 - 4 = 4(x^3 - 1)$$

So $f'(x) = 0$ if $x^3 - 1 = 0$, so $x = 1$, and $f(1) = 1 - 4 + 1 = -2$

3) Compare:

The absolute minimum of f is $f(1) = -2$ and the absolute maximum of f is $f(2) = 9$.

6. *(10 points)* Show that if $f'(x) > 0$ for all x, then f is increasing.

Hint: Assume $b > a$ and show that $f(b) > f(a)$

This is actually much easier than you think!

Assume $b > a$

By the **MVT** on $[a, b]$, we get:

$$\frac{f(b) - f(a)}{b - a} = f'(c) > 0$$

So:

$$\frac{f(b) - f(a)}{b - a} > 0$$

Hence, multiplying by $b - a > 0$, we get:

$$f(b) - f(a) > 0$$

So

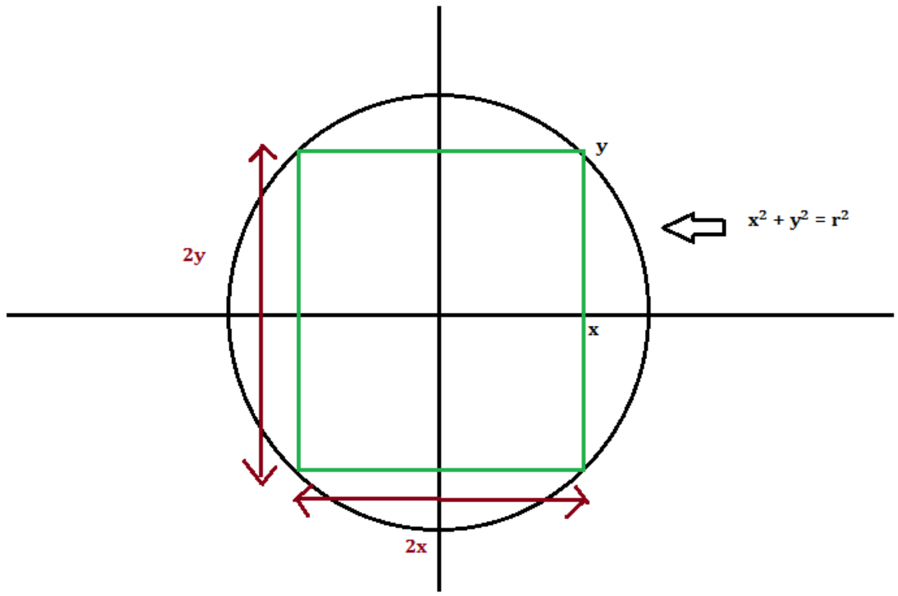
$$f(b) > f(a)$$

And we're done!

7. *(20 points)* Find the dimensions of the rectangle of largest area that can be inscribed in (put inside of) a circle of radius 1.

1) First draw a good picture! (here $r = 1$)

1A/Practice Exams/Mockrectangle.png



2) Based on your picture, the length of the rectangle is $2x$ and the width is $2y$, and the area is $A = (2x)(2y) = 4xy$. But since (x, y) is on the circle, $x^2 + y^2 = 1$, so $y = \sqrt{1 - x^2}$, so $A(x) = 4x\sqrt{1 - x^2}$. But instead of maxizing A, let's maximize:

$$f(x) = A^2 = 16x^2(1 - x^2) = 16x^2 - 16x^4$$

3) The constraint is $0 \leq x \leq 1$

4) $f'(x) = 32x - 64x^3 = 32x(1 - 2x^2) = 0 \Leftrightarrow x = 0 \text{or} x^2 = \frac{1}{2} \Leftrightarrow x = \frac{1}{\sqrt{2}}$ (we're ignoring $x = 0$ and $x = -\frac{1}{\sqrt{2}}$ here.
Also, $f(0) = f(r) = 0$, and $f(\frac{1}{\sqrt{2}}) > 0$ (we don't really care what it is, as long as it's positive), so by the closed interval method, $x = \frac{1}{\sqrt{2}}$ is an absolute maximizer.

So our answer is:

$$Length = 2x = \sqrt{2}, Width = 2y = 2\sqrt{1 - \frac{1}{2}} = 2\frac{1}{\sqrt{2}} = \sqrt{2} = x$$

So the optimal rectangle is a **square!!!**

8. (**ONLY** do this one if you got completely stuck on problem 7.)

If 12 cm^2 of material is available to make a box with a square base and an open top, find the largest volume of the box.

1) Draw a picture of a box without the top, label the sides of the base r and the height h.

2) The volume is $V = r^2 h$. However, the surface area of the box without the top is $S = r^2 + 4rh = 12$, so $h = \frac{12 - r^2}{4r} = \frac{3}{r} - \frac{r}{4}$, whence:

$$V(r) = r^2 h = r^2 \left(\frac{3}{r} - \frac{r}{4} \right) = 3r - \frac{r^3}{4}$$

3) <u>Constraint:</u> $r > 0$

4)
$$V'(r) = 3 - \frac{3}{4}r^2 = 0 \Leftrightarrow \frac{3}{4}r^2 = 3 \Leftrightarrow r^2 = 4 \Leftrightarrow r = 2$$

By FDTAEV, $r = 2$ maximizes the volume of the box.

And if $r = 2$, $h = \frac{3}{r} - \frac{r}{4} = \frac{3}{2} - \frac{1}{2} = 1$, and so the largest volume is $\boxed{V = r^2 h = 4}$

Bonus 1 *(5 points)* Show that $f(x) = \ln(x) - x$ does not have a slant asymptote at ∞.

Hint: Assume $f(x)$ has a slant asymptote $y = mx + b$ at ∞. Calculate m, then calculate b, and find a contradiction!

Following the hint, let's calculate m:

$$m = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{\ln(x) - x}{x} \lim_{x\to\infty} \frac{\ln(x)}{x} - 1 = -1$$

And

$$b = \lim_{x\to\infty} f(x) - (mx) = \lim_{x\to\infty} \ln(x) - x + x = \lim_{x\to\infty} \ln(x) = \infty$$

This is a contradiction (b has to be finite), hence f does not have a S.A. at ∞

Bonus 2 *(5 points)*

Assume $-1 < f(x) < 1$ and $f'(x) \neq 1$ for all x. Show that f has exactly one fixed point.

At least one fixed point: Let $g(x) = f(x) - x$. Then $g(-1) = f(-1) + 1 > 0$ (since $f(x) > -1$) and $g(1) = f(1) - 1 < 0$ (since $f(x) < 1$), so by the IVT, g has at least one zero, so f has at least one fixed point.

At most one fixed point: Assume f has 2 fixed points a and b, then $f(a) = a$ and $f(b) = b$. Then, by the MVT on $[a, b]$:

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

But $f(b) = b$ and $f(a) = a$, so:

$$\frac{b - a}{b - a} = f'(c)$$

So:

$$1 = f'(c) \neq 1$$

Which is a contradiction, and hence we're done!